LEONES ACADEMICA

Gauss's Clock: Mastering Modular Arithmetic and Divisibility Tricks

Modular arithmetic, often called "clock arithmetic", was pioneered by the brilliant mathematician Carl Friedrich Gauss in 1801. This new approach, which we will explore, revolutionized the way mathematicians thought about numbers and their properties. Imagine transforming complex calculations into a game of numbers, where the results wrap around a fixed set of values, just like the hours on a clock. Gauss's introduction of this concept was like adding a new tool to the mathematical toolbox, making it possible to solve problems related to divisibility and congruences with greater ease and elegance. His work laid the foundation for modern number theory, influencing countless mathematical discoveries and applications.

Think about a standard 12-hour clock. When the hour hand moves past 12, it doesn't go to 13 but resets to 1. This simple mechanism is a perfect way to understand modular arithmetic. If it's 9 o'clock now and we want to know what time it will be in 7 hours, we add 9 and 7 to get 16. However, clocks don't have a 16, so we loop back to 4. In mathematical terms, 16 (mod 12) $\equiv 4$. Here, 12 is the modulus, and this cyclical pattern is at the heart of modular arithmetic. Visualizing numbers on a clock helps make the abstract concept of modular arithmetic concrete and accessible. Just like a clock resets after 12, numbers reset after reaching the modulus, creating a fascinating and intuitive pattern that's both easy to understand and incredibly useful.

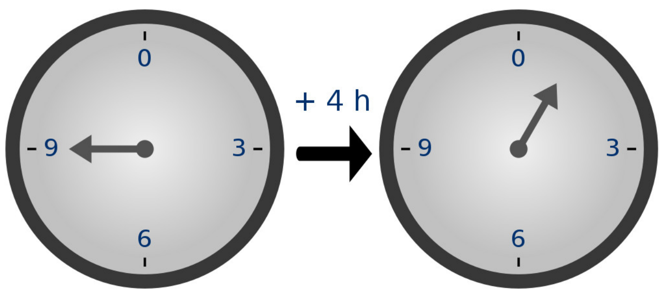

Figure 1: Here we see from a clock that $9 + 4 \pmod{12} \equiv 1$ (Source: ThatsMaths)

Test Your Understanding!

Attempt the following questions. (Hint, if stuck draw a clock with the modulus as the number of digits)

a) $10 + 7 \pmod 5$ b) $5 + 8 \pmod 2$ c) $7 \times 5 \pmod 9$

Different modului have different properties. The modulus 9 in particular has some magical properties that make it particularly interesting and useful in mathematics. One of the coolest features of $\pmod 9$ arithmetic is the digit sum property. If you take any number and add its digits together, the result is congruent to the original number modulo 9. For instance, take the number 258. Adding its digits $(2+5+8)$ gives 15, and adding 1 and 5 gives 6, which is the same as $258 \pmod 9$. $\pmod 9$ arithmetic also simplifies many calculations and reveals fascinating patterns in numbers, making it a powerful tool for mathematicians. Let's explore other moduli to see if we can prove the divisibility tricks for numbers.

Firstly, we need to notice that any number, for example 249 can be written as $9+(4\times 10)+(2\times 10^2)$. If we try to extend this to any number, N we can write the following:

$$N = a_0 + a_1 \times 10 + a_2 \times 10^2 + \dots a_n \times 10^n$$

We can think of this as a_0 being our units digit, a_1 being the tens digit, and a_2 being the hundreds digit, and so on. Finally, before we start we just need to note that taking the modulus of any two numbers multiplied by each other is the same as taking the mod of each number individually and multiplying them. For example:

$$3 \times 5 \pmod 2 \equiv 15 \pmod 2 \equiv 1$$

but we can also do this calculation as:

$$3 \times 5 \pmod 2 \equiv (3 \pmod 2 \times 5 \pmod 2) \pmod 2 \equiv 1 \times 1 \pmod 2 \equiv 1$$

It is often useful for us to approach a problem this way. Also note we use $\equiv$ instead of $=$, but this is just a technicality in modular arithmetic to show equivalence.

Using some of our techniques, we can have a look at some divisibility tests. We will start with some simple ones and work to see some harder tests.

- The test for seeing a number is divisible by 2 is simple. We know that if the units digit is $0, 2, 4, 6$ or 8 it must be the case, so let's prove it! We want to see when $N \pmod 2 \equiv 0$, so following the logic we get:

$$\begin{aligned}
N \pmod 2 &\equiv a_0 + a_1 \times 10 + a_2 \times 10^2 + \dots + a_n \times 10^n \pmod 2 \\
&\equiv \left[a_0 \pmod 2 + a_1 \times 10 \pmod 2 + a_2 \times 10^2 \pmod 2 + \dots + a_n \times 10^n \pmod 2\right] \pmod 2 \\
&\equiv [a_0 \pmod 2 + a_1 \times 0 \pmod 2 + a_2 \times 0 \pmod 2 + \dots + a_n \times 0 \pmod 2] \pmod 2 \\
&\equiv a_0 \pmod 2
\end{aligned}$$

  So we see that a number N is divisible by 2 if its first digit is divisible by 2, i.e. if its first digit is $0, 2, 4, 6$ or 8.

- We will go through the test for divisibility by 3 a bit faster. You may know that any number whose sum of digits is divisible by 3 is also divisible by 3. Let's prove this:

$$\begin{aligned}
N \pmod 3 &\equiv a_0 + a_1 \times 10 + a_2 \times 10^2 + \dots a_n \times 10^n \pmod 3 \\
&\equiv \big[a_0 \pmod 3 + a_1 \times 10 \pmod 3 + a_2 \times 10^2 \pmod 3 + \dots + a_n \times 10^n \pmod 3\big] \pmod 3 \\
&\equiv [a_0 \times 1 \pmod 3 + a_1 \times (9+1) \pmod 3 + a_2 \times (99+1) \pmod 3 + \dots \\
&\quad + a_n \times (10^n - 1 + 1) \pmod 3] \pmod 3 \\
&\equiv a_0 + a_1 + a_2 + \dots + a_n \pmod 3
\end{aligned}$$

In this proof we showed that $10^k \pmod 3 \equiv 1$ using the fact that we know $10^k - 1 \pmod 3 = 0$ since it will just be a string of 9's.

- Our final divisibility test is a less known one: Divisibility for 11. This actually has two different answers depending on the length of our number N. We first need to notice that $10^k \pmod{11} \equiv \pm 1$, where it is -1 if k is odd and 1 if k is even. An example to see this is $10 \pmod{11} \equiv 11 - 1 \pmod{11} \equiv -1$ and similarly $100 \pmod{11} \equiv 99 + 1 \pmod{11} \equiv 1$. Using this, let's find the divisibility test.

$$\begin{aligned}
N \pmod{11} &\equiv a_0 + a_1 \times 10 + a_2 \times 10^2 + \dots a_n \times 10^n \pmod{11} \\
&\equiv \big[a_0 \pmod{11} + a_1 \times 10 \pmod{11} + a_2 \times 10^2 \pmod{11} + \dots \\
&\quad + a_n \times 10^n \pmod{11}\big] \pmod{11} \\
&\equiv a_0 - a_1 + a_2 + \dots \pm a_n \pmod{11}
\end{aligned}$$

The final $\pm a_n$ depends on if n is even or odd. This may be a surprising result, so we will do a quick example. 105204 is divisible by 11 because

$$[(4 + 2 + 0) - (0 + 5 + 1)] \pmod{11} \equiv 0 \pmod{11}$$

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We've seen just one application of modular arithmetic but is this all it is used for? In our digital age, modular arithmetic is the secret behind many technologies that power our world. One of the best applications is in cryptography, the art of encoding information to keep it secure. Every time you send a text, make an online purchase, or access your bank account, modular arithmetic is working behind the scenes to keep your data safe. It's used in algorithms like RSA encryption, which relies on the properties of large prime numbers and modular arithmetic to create unbreakable codes. But that's not all – modular arithmetic is also crucial in computer science for creating efficient algorithms, managing some data structures, and even in error detection systems to ensure data integrity. Whether it's the clock on your phone, the rhythm in a song, or the security of your online transactions, modular arithmetic is everywhere, making our lives easier and more secure.

Challenge

Attempt the following questions, using what we have learned

a) Is 9020814 divisible by 11? b) Prove the divisibility test for 5

c) Prove the divisibility test for 13 (Hint: the answer is take the number, remove the unit digit and add 4 times the unit digit then repeat)

d) Test your rule on 59371