Secrets of the Fibonacci Sequence: The Golden Ratio

Introduction

Mathematics is full of beautiful patterns, and two of the most fascinating are the Fibonacci Sequence and the Golden Ratio. These patterns appear in nature, art, and architecture, making them not just theoretical curiosities but also practically significant. Let's explore these two concepts, understand their significance, and discover how they are related.

The Fibonacci Sequence

The Fibonacci Sequence is named after the Italian mathematician Leonardo of Pisa, who was known as Fibonacci. He introduced this sequence to Western mathematics in his 1202 book Liber Abaci (The Book of Calculation). The book was significant because it helped popularize the Arabic numeral system (the one we use today) in Europe (the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ in place of the Roman numeral system).

Fibonacci posed a problem about the growth of a population of rabbits, assuming ideal conditions. The solution to this problem was the sequence we now know as the Fibonacci Sequence. Although the sequence was known to Indian mathematicians as early as the 6th century, Fibonacci's work brought it into the mainstream of European mathematics.

Month	1	2	3	4	5	6	7	8
New born pairs	1	0	1	1	2	3	5	8
Month old pairs	0	1	0	1	1	2	3	5
Breeding pairs	0	0	1	1	2	3	5	8
Total	1	1	2	3	5	8	13	21

Figure 1: Fibonacci's Rabbit problem. Starting with one pair of rabbits, he observes how many pairs of rabbits we eventually get, assuming rabbits only breed after 2 months of birth. (Source: Wade Deacon High School)

The Fibonacci Sequence is a simple yet intriguing sequence of numbers that starts with 0 and 1. Each subsequent number is the sum of the two preceding numbers. Here's how it looks:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Mathematically, the Fibonacci Sequence can be defined as:

$$F_n = F_{n-1} + F_{n-2}$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. This sequence grows rapidly, and you'll notice that each number is roughly 1.618 times larger than the one before it as the sequence progresses. This number, 1.618..., is no coincidence—it's the Golden Ratio!

The Golden Ratio

The Golden Ratio, denoted by the Greek letter ϕ (phi), is a special number approximately equal to 1.61803398875, but goes on forever. It has fascinated mathematicians, artists, and architects for centuries due to its aesthetic properties. The Golden Ratio is often associated with beauty, as many believe that objects and artworks that follow the proportion ϕ are more visually appealing.

The Golden Ratio has a much richer history, stretching back to ancient Greece. The first known use of the Golden Ratio dates to around 300 BCE, in the work of the famous Greek mathematician Euclid, who described it in his ground–breaking treatise Elements. Euclid called it the "extreme and mean ratio", but the term "Golden Ratio" wasn't coined until much later.

The Golden Ratio became particularly significant in the Renaissance, where it was known as the "Divine Proportion" because it was believed to embody a sense of perfection and harmony. Artists and architects, such as Leonardo da Vinci, used the Golden Ratio in their works to achieve aesthetically pleasing compositions.

The association of the Golden Ratio with beauty and harmony has persisted into modern times, making it a subject of interest not only in mathematics but also in art, architecture, and nature studies.

Defining the Golden Ratio

The Golden Ratio ϕ is defined mathematically by the following equation:

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

This equation arises from the solution to the quadratic equation:

$$x^2 - x - 1 = 0$$

Which is derived from making the ratios $a + b : a$ and $a : b$ identical. This equation has two solutions, ϕ and its conjugate $\bar{\phi} = \frac{1-\sqrt{5}}{2}$, which is approximately -0.618. While ϕ is positive and has special significance, its negative counterpart is less commonly discussed.

The Silver Ratio and Other Metallic Ratios

Aside from the Golden Ratio, there are other ratios derived from similar quadratic equations. For example, the Silver Ratio, often denoted by δ_s, is approximately 2.414 and is the solution to the equation:

$$x^2 - 2x - 1 = 0$$

These types of ratios are collectively known as *metallic ratios*, with each ratio having its unique properties and applications.

Test Your Understanding!

Attempt the following questions.

a) Can you think of more assumptions Fibonacci used for his rabbit problem?

b) Try to prove $\phi = 1 + \frac{1}{\phi}$ c) What is ϕ^2?

The Link Between the Fibonacci Sequence and the Golden Ratio

So, how do these two concepts connect? As we look further into the Fibonacci Sequence, an interesting pattern emerges. If you take the ratio of consecutive Fibonacci numbers, $\frac{F_{n+1}}{F_n}$, you will find that as n increases, this ratio approaches the Golden Ratio ϕ.

Mathematical Explanation

Let's denote the ratio of consecutive Fibonacci numbers as r_n:

$$r_n = \frac{F_{n+1}}{F_n}$$

If we compute these ratios for increasing values of n:

$$\frac{1}{1} = 1, \quad \frac{2}{1} = 2, \quad \frac{3}{2} \approx 1.5, \quad \frac{5}{3} \approx 1.6667, \quad \frac{8}{5} = 1.6, \quad \frac{13}{8} = 1.625$$

Notice how these ratios are getting closer and closer to 1.61803398875, the value of ϕ. In fact, as n tends to infinity:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$$

More formally, this happens because if we take $F_{n+1} = F_n + F_{n-1}$, and $\lim_{n\to\infty} r_n = L$, we see by diving the Fibonacci rule by F_n, we have $\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$. Taking limits as $n \to \infty$ we immediately see that:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n} \implies L = 1 + \frac{1}{L}$$

Rearranging this we get our original quadratic and find $L = \phi$! The recursive definition of the Fibonacci Sequence mirrors the algebraic definition of the Golden Ratio. As the numbers in the sequence grow, the influence of the initial conditions (0 and 1) diminishes, and the ratio of successive terms stabilizes at ϕ.

Conclusion

The Fibonacci Sequence and the Golden Ratio are two of mathematics' most famous concepts, and their relationship is a beautiful example of how different mathematical ideas can be intertwined. The fact that the ratio of consecutive Fibonacci numbers converges to the Golden Ratio as n increases is not just a numerical coincidence but a deep and intrinsic connection that appears in many places in mathematics and nature.

Understanding these connections gives us a greater appreciation of the elegance of mathematics. Whether you encounter these concepts in a mathematical proof, a piece of art, or the structure of a flower, you'll know that you're seeing the beautiful interplay between numbers and nature.

Challenge

Have a go at these to push your knowledge!

a) If instead $F_0 = a$ and $F_1 = b$, what does the ratio of the terms converge to?

b) Explore the Tribonacci Sequence, $T_0 = 0, T_1 = 1, T_2 = 1$ where $T_n = T_{n-1} + T_{n-2} + T_{n-3}$